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LETTER TO THE EDITOR

n-dimensional equations with the maximum number of symmetry generators

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Abstract. In this letter we obtain the general class of ordinary differential equations that can be reduced by a point transformation to the free-particle equation and give the general form of the Lie symmetry generators for these equations. We apply these results to obtain the symmetry generators for the n-dimensional harmonic oscillator and for a particle in a constant magnetic field.

The utilisation of a coordinate transformation to reduce one differential equation to another differential equation with a known solution is an old procedure, dating from the very beginning of the development of the differential calculus. In this context the problem of the linearisation of a differential equation has paramount importance. In general, linearisation amounts to finding necessary and sufficient conditions for the equation to be locally equivalent to the free-particle equation. In recent years this procedure has been employed by several authors in the analysis of the symmetry structures of second-order differential equations and, in quantum mechanics, for obtaining of propagators starting from the free-particle propagator (see, for example, Junker and Inomata 1985).

In a recent letter we obtained the general class of one-dimensional equations that can be reduced to the free-particle equation by an invertible point transformation (Duarte *et al* 1987). We also found the general form of their Lie symmetry generators, which have a Lie algebra isomorphic to the free-particle symmetry group, SL(3, R). The same problem, in the one-dimensional case, has been investigated independently by Sarlet *et al* (1987) (see also Leach and Mahomed 1985). González-López (1988a, b) analysed those n-dimensional linear systems which have the maximal group of Lie symmetries, SL($n+2$, R) and are reduced to the n-dimensional free-particle equation. It is also worth observing that Cartan considered this problem, as long ago as 1924 (reprinted in Cartan 1984), from a geometrical point of view. Here we extend, for n-dimensional systems, the results of our previous letter. We obtain the general class of ordinary differential equations which can be reduced by a point transformation to the free-particle equations and give the general form of the Lie symmetry generators for this system. The same point transformation can be used to find a Lagrangian for this class of equations, starting from the free-particle Lagrangian. We also consider the application of these results to the identification of the symmetry generators for the n-dimensional harmonic oscillator and to the case of a particle in a constant magnetic field.

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If we start from the n-dimensional free-particle equations

$$\mathrm{d}^2X^i/\mathrm{d}T^2=0 \tag{1}$$

and make an invertible point transformation

$$\begin{aligned} X^i&=F^i(x^j,t) \qquad & x^i&=P^i(X^j,T)\\ T&=G(x^j,t) \qquad & t&=Q(X^j,T)\end{aligned} \tag{2}$$

we get the following class of equations:

$$\Delta^i_k\ddot{x}^k+\Lambda^i_{jkl}\dot{x}^j\dot{x}^k\dot{x}^l+\Gamma^i_{jk}\dot{x}^j\dot{x}^k+V^i_j\dot{x}^j+L^i=0 \tag{3}$$

where

$$\begin{aligned}\Delta^i_k&=(F^i_{/k}G_{/l}-G_{/k}F^i_{/l})\dot{x}^l+G_{/t}F^i_{/k}-F^i_{/t}G_{/k}\\ \Gamma^i_{jk}&=2G_{/j}F^i_{/tk}+G_{/t}F^i_{/jk}-2F^i_{/j}G_{/tk}-F^i_{/t}G_{/jk}\\ V^i_j&=2G_{/t}F^i_{/tj}+G_{/j}F^i_{/tt}-2F^i_{/t}G_{/tj}-F^i_{/j}G_{/tt}\\ L^i&=G_{/t}F^i_{/tt}-F^i_{/t}G_{/tt}\end{aligned} \tag{4}$$

with $i, j, k, l=1,\ldots,n$ and $F^i_{/k}=\partial F^i/\partial x^k$. The condition which must be satisfied if the transformations (2) are to be invertible is

$$\det\begin{bmatrix}F^1_{/1} & \cdots & F^1_{/n} & F^1_{/t}\\ \vdots & \ddots & \vdots & \vdots\\ F^n_{/1} & \cdots & F^n_{/n} & F^n_{/t}\\ G_{/1} & \cdots & G_{/n} & G_{/t}\end{bmatrix}\neq 0. \tag{5}$$

The free-particle equations (1) have the following symmetry generators, obtained by using the usual Lie conditions (Wulfman and Wybourne 1976):

$$\begin{aligned} &U_1=\partial/\partial T \qquad U_2=T\partial/\partial T \qquad U^i_3=X^i\partial/\partial T\\ &U^i_4=\partial/\partial X^i \qquad U_5=T^2\partial/\partial T+TX^i\partial/\partial X^i \qquad U^i_6=T\partial/\partial X^i\\ &U^i_7=X^iT\partial/\partial T+X^iX^j\partial/\partial X^j \qquad U^{ij}_8=X^i\partial/\partial X^j\end{aligned} \tag{6}$$

where $1\leqslant i,j\leqslant n$. The symmetry algebra of these generators is the SL $(n+2, R)$ algebra whose dimension is n^2+4n+3.

From (2) and (6) we get the general form for the symmetry generators of the equations (3):

$$\begin{aligned} &U_1=Q_{/T}(x^j,t)\partial/\partial t+P^i_{/T}(x,t)\partial/\partial x^i \qquad U_2=GQ_{/T}\partial/\partial t+GP^i_{/T}\partial/\partial x^i\\ &U^i_3=F^iQ_{/T}\partial/\partial t+F^iP^j_{/T}\partial/\partial x^j \qquad U^i_4=Q_{/i}\partial/\partial t+P^j_{/i}\partial/\partial x^j.\\ &U^i_5=GQ_{/i}\partial/\partial t+GP^j_{/i}\partial/\partial x^j \qquad U^i_{6j}=F^iQ_{/j}\partial/\partial t+F^iP^k_{/j}\partial/\partial x^k\\ &U_7=(G^2Q_{/T}+GF^iQ_{/i})\partial/\partial t+(G^2P^i_{/T}+GF^jP^i_{/j})\partial/\partial x^i\\ &U^i_8=(GF^iQ_{/T}+F^iF^jQ_{/j})\partial/\partial t+(GF^iP^k_{/T}+F^iF^jP^k_{/j})\partial/\partial x^k\end{aligned} \tag{7}$$

where $1\leqslant i,\ j,\ k\leqslant n$. The symmetry algebra of all the systems with the form (3) is isomorphic to SL$(n+2,\ R)$, with the maximal dimension being n^2+4n+3.

Equations (3) can be obtained from a Lagrangian which is constructed by starting from the usual free Lagrangian

$$\mathscr{L}_0 = \tfrac{1}{2}\dot{X}^i\dot{X}^i \tag{8}$$

and applying the point transformation (2). The general form of this Lagrangian will be

$$\mathscr{L} = \frac{1}{2}\left[\frac{\mathrm{d}F^i/\mathrm{d}t}{\mathrm{d}G/\mathrm{d}t}\right]^2 \frac{\mathrm{d}G}{\mathrm{d}t}. \tag{9}$$

If we start from other equivalent Lagrangians for the free-particle equation we also get equivalent Lagrangians for (3). This is a new method for finding equivalent Lagrangians for the system (3).

The Noether symmetry generators for the Lagrangian (9) follow from the symmetry generators for the free Lagrangian $\mathscr{L}_0$ and constitute a symmetry algebra isomorphic to the symmetry algebra of $\mathscr{L}_0$. The following expressions for the $\frac{1}{2}(n^2+3n+6)$ vector fields, obtained by González-López (1988a) in the linear case, apply also in this situation:

$$\begin{array}{lll} V_1 = U_1 & V_2 = U_2 + U^i_{6i} & V_{3'} = U_7 \\ V^i_4 = U^i_4 & V^i_5 = U^i_5 & V^i_{6j} = U^i_{6j} - U^j_{6i}. \end{array} \tag{10}$$

We shall now analyse particular cases of (3). If we impose the point transformation

$$F^i = A^{ik}(t)x^k \qquad G = g(t) \tag{11}$$

then (3) will have the form

$$A^{ik}\ddot{x} + \left[\frac{2\mathrm{d}A^{ik}}{\mathrm{d}t} - A^{ik}\frac{\mathrm{d}^2g/\mathrm{d}t^2}{\mathrm{d}g/\mathrm{d}t}\right]\dot{x}^k + \left[\left(\frac{\mathrm{d}^2A^{ik}}{\mathrm{d}t^2}\right)x^k - \left(\frac{\mathrm{d}A^{ik}}{\mathrm{d}t}\right)x^k\frac{\mathrm{d}^2g/\mathrm{d}t^2}{\mathrm{d}g/\mathrm{d}t}\right] = 0. \tag{12}$$

(i) First we consider the n-dimensional isotropic oscillator. We can easily show that for the choice

$$A^{ik} = \sec(\omega t)\delta^{ik} \qquad g = \tan(\omega t)/\omega \tag{13}$$

we obtain the equations of motion of an n-dimensional isotropic oscillator:

$$\ddot{x}_i = -\omega^2 x_i. \tag{14}$$

The transformations (13) are the generalisation to n dimensions of the Jackiw transformations (Jackiw 1980) for the one-dimensional case. The Lie symmetry generators for these n-dimensional equations can be obtained directly from (13) and (7).

(ii) Now let us examine the application to a charged particle in a constant magnetic field. If we consider the two-dimensional motion of a charged particle in a plane perpendicular to the direction of a constant magnetic field, the equations of motion will be

$$\ddot{x} = 2\omega\dot{y} \qquad y = -2\omega\dot{x} \tag{15}$$

where $\omega = eB/2mc$.

Equations (15) can be obtained from the free-particle equations by using the transformation

$$A^{ik} = \begin{bmatrix} 1 & -\tan(\omega t) \\ \tan(\omega t) & 1 \end{bmatrix} \qquad g = \tan(\omega t)/\omega. \tag{16}$$

This transformation is a combination of the Jackiw transformation and a Larmor rotation. From (16) and (7), we can get the Lie symmetry generators for (15):

$$U_1=\partial/\partial t \qquad U_2=\partial/\partial x \qquad U_3=\partial/\partial y \qquad U_4=y\partial/\partial x-x\partial/\partial y$$

$$U_5=x\partial/\partial x+y\partial/\partial y \qquad U_6=\cos(2\omega t)\partial/\partial x-\sin(2\omega t)\partial/\partial y$$

$$U_7=\sin(2\omega t)\partial/\partial x+\cos(2\omega t)\partial/\partial y$$

$$U_8=-(\cos(2\omega t)/2\omega)\partial/\partial t+x\sin(2\omega t)\partial/\partial x+x\cos(2\omega t)\partial/\partial y$$

$$U_9=-(\sin(2\omega t)/2\omega)\partial/\partial t-x\cos(2\omega t)\partial/\partial x+x\sin(2\omega t)\partial/\partial y$$

$$U_{10}=(\sin(2\omega t)/2\omega)\partial/\partial t+y\sin(2\omega t)\partial/\partial x+y\cos(2\omega t)\partial/\partial y \qquad (17)$$

$$U_{11}=-(\cos(2\omega t)/2\omega)\partial/\partial t-y\cos(2\omega t)\partial/\partial x+y\sin(2\omega t)\partial/\partial y$$

$$U_{12}=-(y/2\omega)\partial/\partial t+\tfrac{1}{2}(x^2-y^2)\partial/\partial x+xy\partial/\partial y$$

$$U_{13}=(x/2\omega)\partial/\partial t+xy\partial/\partial x+\tfrac{1}{2}(x^2-y^2)\partial/\partial y$$

$$U_{14}=[x\sin(2\omega t)+y\cos(2\omega t)]\partial/\partial t+\omega\cos(2\omega t)(x^2+y^2)\partial/\partial x - \omega\sin(2\omega t)(x^2+y^2)\partial/\partial y$$

$$U_{15}=[-x\cos(2\omega t)+y\sin(2\omega t)]\partial/\partial t+\omega\sin(2\omega t)(x^2+y^2)\partial/\partial x + \omega\cos(2\omega t)(x^2+y^2)\partial/\partial y.$$

The Lagrangian obtained from the free Lagrangian by using (16) is

$$\mathscr{L}=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+\omega(x\dot{y}-\dot{x}y)+(\mathrm{d}/\mathrm{d}t)[\tfrac{1}{2}\omega(x^2+y^2)\tan(\omega t)]. \qquad (18)$$

We observe that the equations for an anisotropic oscillator, or for the full three-dimensional motion of a charged particle in a constant magnetic field, do not have the form (3). Therefore, there is no invertible transformation that reduces these equations to the free-particle equations. These are particular cases of the theorem of González-López (1988a), which can be demonstrated from (3) and (4) by imposing $\Lambda^i_{jkl}=0$, $\Gamma^i_{jk}=0$, $V^i_j=V^i_j(t)$ and $L^i=B^{ik}(t)x^k+C^i(t)$. Examples of non-linear equations with the maximal symmetry structure can be obtained directly from (3).

The procedure employed here to find the general class of n-dimensional equations with the maximal Lie symmetry structure, i.e. that can be reduced to the free equation, can be generalised for other classes of equations with a different symmetry group. For example, we can start from the equations of motion for the three-dimensional Kepler problem and find the class of equations that can be transformed, by (2), into these equations. We thus have a method for generating classes of integrable equations and for finding their Lie symmetry groups if we start from integrable equations with a known symmetry structure. It is useful to extend the same procedure to partial differential equations which can be transformed, for example, into the free wave equation.

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